

Crossover from droplet to flat initial conditions in the KPZ equation from the replica Bethe ansatz

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Abstract. We show how our previous result based on the replica Bethe ansatz for the Kardar Parisi Zhang (KPZ) equation with the "half-flat" initial condition leads to the Airy_2 to Airy_1 (i.e. GUE to GOE) universal crossover one-point height distribution in the limit of large time. Equivalently, we obtain the distribution of the free energy of a long directed polymer (DP) in a random potential with one fixed endpoint and the other one on a half-line. We then generalize to a DP when each endpoint is free on its own half-line. It amounts, in the limit of large time, to obtain the distribution of the maximum of the transition process $\text{Airy}_{2 \rightarrow 1}$ (minus a half-parabola) on a half line.

1. Introduction

Recently there was a lot of progress in finding exact solutions to the one-dimensional noisy Kardar-Parisi-Zhang (KPZ) equation. This equation [1] describes the growth of an interface, in the continuum, parameterized by its height field $h(x, t)$ at point x and has numerous experimental realizations [2, 3]. The growth is generated by an additive space-time white noise, and the problem is to characterize the statistics of the height field as a function of time t . While the scaling exponents $h \sim t^{1/3}$, $x \sim t^{2/3}$ have been known for a while [4], the recent focus is on the PDF (probability distribution function) of the height field. The KPZ problem can be mapped to the problem of a continuous directed polymer (DP) in a quenched random potential, in such a way that $h(x, t) = \ln Z(x, t)$ is proportional to the free energy of the DP with endpoint at x and length t .

As was anticipated from exact solutions of discrete models which belong to the KPZ universality class, such as the PNG growth model [5, 6, 7], the TASEP particle transport model [8, 9, 10] or discrete DP models [11, 6], one expects only a few universal statistics at large time, depending on the type of initial condition. Remarkably, the interface retains some memory of the initial condition, even at large time.

For the KPZ equation on the infinite line there are three main classes. The *droplet initial condition* (which corresponds to a DP with two fixed endpoints) leads to height fluctuations governed at large time by the Tracy Widom (TW) distribution F_2 , the CDF (cumulative distribution function) of the largest eigenvalue of the GUE random matrix ensemble [12]. It was solved simultaneously by two methods. The first route uses a limit from an ASEP model with weak asymmetry [13] and has allowed for a rigorous derivation [14, 15]. The second route [16, 17] uses methods of disordered systems, namely replica, and methods from integrable systems, namely the Bethe Ansatz. It works on the DP version and allows to calculate the integer moments of $Z = e^h$ from the known exact solution of the Lieb-Liniger delta Bose gas [18]. Extracting the PDF for h from the integer moments is, as yet, a non-rigorous step. For the droplet initial conditions, both methods obtained the CDF for all times t , in the form of a Fredholm determinant, nicely displaying convergence to F_2 as $t \rightarrow +\infty$.

The second important class, *the flat initial condition*, was solved using the Replica Bethe Ansatz (RBA) [19, 20, 21] in the form of a Fredholm Pfaffian, valid at all times. At large time the CDF of the height converges to the TW distribution F_1 associated with the GOE ensemble of random matrices. A rigorous derivation is only presently in progress [22]. In fact, remarkable developments have occurred in the math community, from the study of the so-called q -TASEP and related models, which aim to produce many rigorous results as limit processes (e.g. as $q \rightarrow 1$) [23, 24].

The RBA also allowed for the solution of the last important class, *the stationary*

KPZ [25], and of the KPZ equation on the half-line [26] which relates to the GSE random matrix ensemble. Note that all the above mentioned exact solutions arise due to an emerging, and somewhat miraculous, Fredholm determinant or pfaffian structure, found to hold for arbitrary time. Another important aim is to use the RBA to derive systematically the large time asymptotics, even when the knowledge of the finite time result is unavailable. This strategy was recently explored, leading to another set of results [27, 28, 29, 30, 31, 32, 33]. The joint distribution of $h(x, t)$ at several space points was obtained [28, 29, 30, 31]. This is not strictly a "new" result since from the PNG and TASEP models it was anticipated that the (scaled) many point statistics of $h(x, t)$ converge to the one of the Airy_2 process $\mathcal{A}_2(x)$ (minus a parabola) whose one-point CDF is given by F_2 . The Airy_2 process may be defined as the trajectory of the largest eigenvalue of the GUE Dyson Brownian motion (for definition and review see e.g. [29, 9], see also [34]). However, recovering this result within the RBA is a non-trivial and interesting result. Another breakthrough was the calculation within the RBA of the endpoint distribution of the DP [32] directly for infinite time, which was found to agree with the (simultaneous) result about the position of the maximum of the Airy 2 process (minus a parabola) [35, 36, 37, 38]. In both cases, the corresponding finite time problem is unsolved and seems very difficult. A genuinely new result is the recent calculation from RBA [33] of the two-time distribution for the KPZ equation at infinitely separated times. It is important to note that the manipulations leading directly to the infinite time limit in the RBA involve a substantial amount of "guessing" which makes it even less rigorous. However, from the point of view of heuristics it is a very interesting route to explore further.

Besides these three main classes, one also expects three universal *crossover classes* (also called transition classes) with initial conditions which interpolate from one of the three classes at $x = -\infty$ to a distinct one at $x = +\infty$, see e.g. Fig. 4 in Ref. [15]. The aim of the present paper is to study the transition from GUE to GOE statistics in the KPZ equation. This is realized for the so-called "half-flat" initial condition, which is flat to the left and droplet-like to the right. Interestingly, in Ref. [19, 20] we had already obtained the formula for the moments $\overline{Z^n}$ for the half-flat initial condition. There we studied only the $x \rightarrow -\infty$ limit of this formula to solve the flat case for arbitrary time. This formula seems hard to analyze for arbitrary time, however here we consider its large time limit and obtain the PDF for the KPZ height in the form of a Fredholm determinant interpolating between the F_2 and F_1 distributions. We obtain a new closed formula for the Kernel and shows that it is equivalent, via some Airy function identities, with the one obtained in Appendix A of Ref. [39] from a solution of the TASEP. The corresponding Airy process was defined and characterized there and called $\mathcal{A}_{2 \rightarrow 1}$.

Since the recipe for the large time limit seems to work, we extend the calculation to obtain a genuinely new result. We consider the DP problem in the situation where

each endpoint is free on its own half-line. It can again be solved in terms of Fredholm determinants, with new Kernels. Recast in terms of Airy processes, it amounts to obtain the distribution of the maximum of the transition process $\text{Airy}_{2 \rightarrow 1}$ (minus a half-parabola) on a half line.

The outline of the paper is as follows. In Section 2 we recall the KPZ and DP models and their connection and define the dimensionless units. In Section 3 we recall the known results for the droplet and the flat initial conditions. In Section 4 we explain what we aim to do in this paper, give elementary facts about the transition process $\text{Airy}_{2 \rightarrow 1}$ and introduce the generating function. In Section 5 we briefly recall the replica Bethe ansatz method and in Section 6, the formula from Ref. [19, 20]. In Section 8 we consider the large time limit of this formula, and obtain the new form for the Kernel of the transition process. In Section 9 we use identities between Airy functions to put it in a form which is then compared in Section 10 to the previous results of Ref. [39]. Finally in Section 11 we generalize the problem, obtain the new Kernels for the so-called LL and LR problems and explain the connection to the extrema of the $\text{Airy}_{2 \rightarrow 1}$ process. We conclude on open problems and give some details in the Appendices.

2. Model and dimensionless units

2.1. the KPZ equation

Consider the standard 1D continuum KPZ growth equation for the height field $h(x, t)$:

$$\partial_t h = \nu \nabla^2 h + \frac{1}{2} \lambda_0 (\nabla h)^2 + \sqrt{D} \eta \quad (1)$$

in presence of the white noise $\overline{\eta(x, t)\eta(x', t')} = \delta(x - x')\delta(t - t')$. We define the scales

$$x_0 = (2\nu)^3, \quad t_0 = 2(2\nu)^5, \quad \lambda_0 h_0 = 2\nu \quad (2)$$

and use them as units, i.e. we set $x \rightarrow x_0 x$, $t \rightarrow t_0 t$ and $h \rightarrow h_0 h$ and work from now on in the dimensionless units where the KPZ equation becomes:

$$\partial_t h = \nabla^2 h + (\nabla h)^2 + \sqrt{2\bar{c}} \eta \quad (3)$$

$$\bar{c} = D\lambda_0^2 \quad (4)$$

2.2. the directed polymer

Consider now $Z(x, t|y, 0)$ the partition function of the continuum directed polymer in the random potential $-\sqrt{\bar{c}} \eta(x, t)$ with fixed endpoints at (x, t) and $(y, 0)$, at temperature T :

$$Z(x, t|y, 0) = \int_{x(0)=y}^{x(t)=x} Dx e^{-\frac{1}{T} \int_0^t d\tau [\frac{1}{2} (\frac{dx}{d\tau})^2 - \sqrt{\bar{c}} \eta(x(\tau), \tau)]} \quad (5)$$

As is well known it can be mapped onto the KPZ equation with the correspondence:

$$\frac{\lambda_0}{2\nu}h \equiv \ln Z \quad , \quad T = 2\nu \quad (6)$$

Here and below, overbars denote averages over the white noise η . For both problems we define, as in Ref. [16, 17, 19] the dimensionless parameter $\sim t^{1/3}$:

$$\lambda = \frac{1}{2}(\bar{c}^2 t / T^5)^{1/3}, \quad (7)$$

which measures the scale of the fluctuations of the DP free energy, i.e. of the KPZ height. From now on we use the same units $x_0 = T^3$ and $t_0 = 2T^5$ as above, and in these dimensionless units the partition sum $Z = Z(x, t|y, 0)$ is the solution of:

$$\partial_t Z = \nabla^2 Z + \sqrt{2\bar{c}} \eta Z \quad (8)$$

with initial condition $Z(x = 0, t|y, 0) = \delta(x - y)$. In these units the dimensionless parameter is $\lambda = (\bar{c}^2 t / 4)^{1/3}$.

2.3. Cole-Hopf mapping

The Cole-Hopf mapping solves the KPZ equation in terms of the DP partition sum, in the dimensionless units:

$$e^{h(x,t)} = \int dy Z(x, t|y, 0) e^{h(y,t=0)}. \quad (9)$$

which maps Eq. (3) into (8). We will thus also adopt the notation:

$$h(x, t|y, 0) = \ln Z(x, t|y, 0) \quad (10)$$

although it is somewhat improper since it requires a regularization near $t = 0$ (see below).

Below, when specified, we will often set $\bar{c} = 1$ which amounts to a further change of units $x \rightarrow x/\bar{c}$ and $t \rightarrow t/\bar{c}^2$.

3. Known results for the droplet and flat initial conditions

The droplet initial conditions for the KPZ equation is by definition the narrow wedge:

$$h_{drop}(x, t = 0) = -w|x| \quad , \quad w \rightarrow +\infty \quad (11)$$

and corresponds to the fixed endpoint initial condition $Z(x, t = 0) = \delta(x)$ for the DP. More generally $h(x, t|y, 0) \equiv h_{drop}(x - y, t) + \ln(\frac{w}{2})$ given by (10) corresponds to a sharp wedge centered in y . Everywhere in this paper \equiv means equivalent *in law*. The additive constant $\ln(\frac{w}{2})$ is necessary for regularization, but we will ignore below all time-independent constants.

It is known [13, 14, 15, 16, 17] that at large time the one-point fluctuations of the height grow as $t^{1/3}$ and are governed by the GUE Tracy Widom (cumulative) distribution $F_2(s)$ as:

$$h_{drop}(0, t) = \ln Z_{drop}(0, t) \simeq v_0 t + \bar{c}^{2/3} t^{1/3} \chi_2 \quad (12)$$

$$\text{Prob}(\chi_2 < s) = F_2(s) = \text{Det}[I - P_0 K_{Ai}^s P_0] \quad (13)$$

where $F_2(s)$ is given by a Fredholm determinant with the Airy Kernel:

$$K_{Ai}^s(v_1, v_2) = K_{Ai}(v_1 + s, v_2 + s) \quad , \quad K_{Ai}(v, v') = \int_{y>0} dy Ai(y+v) Ai(y+v') \quad (14)$$

and $P_0(v) = \theta(v)$ is the projector on R^+ .

More generally, for droplet initial conditions, the multi-point correlation is believed to converge [29, 15, 40] to the ones of the Airy_2 process $\mathcal{A}_2(u)$ [6, 9] with (in units \ddagger where $\bar{c} = 1$):

$$h(x, t) \simeq v_0 t + t^{1/3} (\mathcal{A}_2(u) - u^2) \quad , \quad u = \frac{x}{2t^{2/3}} \quad (15)$$

where $\mathcal{A}_2(0) \equiv \chi_2$ and the process $\mathcal{A}_2(u)$ is stationary, i.e. statistically translationally invariant in u .

For the flat initial condition $h(x, t=0) = 0$, it was found [19, 20] that \S :

$$h_{flat}(0, t) = \ln Z_{flat}(0, t) = v_0 t + \bar{c}^{2/3} 2^{-2/3} t^{1/3} \chi_1 \quad (16)$$

$$\text{Prob}(\chi_1 < s) = F_1(s) = \text{Det}[1 - P_0 B_s P_0] \quad (17)$$

where $F_1(s)$ is the GOE Tracy Widom (cumulative) distribution which is expressed as a Fredholm determinant with the Kernel $B_s(v, v') = Ai(v+v'+s)$. Again it is believed that in that case the joint distribution of the heights $\{h_{flat}(x, t)\}_x$ are governed by the so-called Airy_1 stationary process $\mathcal{A}_1(u)$ (switching back to units where $\bar{c} = 1$):

$$h(x, t) \simeq v_0 t + 2^{1/3} t^{1/3} \mathcal{A}_1(2^{-2/3} u) \quad (18)$$

where $\mathcal{A}_1(0) = \frac{1}{2} \chi_1$. The Airy_1 process is related to the largest eigenvalue of the GOE Dyson Brownian motion. For definition and normalizations see e.g. Ref. [10, 9, 38].

Note that there is a connection between these results. Indeed from the definition one expects, in the large time limit:

$$h_{flat}(x, t) = \ln \int dy e^{h(x, t|y, 0)} \equiv \ln \int dy e^{h(y, t|x, 0)} \simeq v_0 t + t^{1/3} \max_u (\mathcal{A}_2(u) - u^2) \quad (19)$$

\ddagger In several works, e.g. [15, 29, 38], the dimensionless KPZ equation is defined as (1) with $\nu = \frac{1}{2}$, $\lambda_0 = 1$ and $D = 1$. Compared to our dimensionless problem with $\bar{c} = 1$, this is equivalent to only a change of the time: let us denote the time t' there, then $t' = 2t$ where t denotes the time here. We chose to conserve all notations and conventions of our previous works.

\S Here and above v_0 has a non-universal part depending on the regularization of the model at short scale. However, as detailed in [21] if one considers $\ln(Z(t)/\bar{Z}(t))$ then it is universal with $v_0 = -\bar{c}^2/12$ in our units

where we have used that the sets $\{h(x, t|y, 0)\} \equiv \{h(y, t|x, 0)\}$ are statistically equivalent and that, since height fluctuations grow as $t^{1/3}$, the integral is dominated by the maximum. Hence:

$$\max_u (\mathcal{A}_2(u) - u^2) = 2^{-2/3} \chi_1 = 2^{1/3} \mathcal{A}_1(0) \quad (20)$$

i.e. the maximum of the Airy_2 process minus a parabola is given by the Airy_1 process at one point, as proved in [38].

4. Aim of this paper: crossover from droplet to flat

4.1. half-flat initial conditions and STS identity

Consider the double wedge initial condition for the KPZ field on the real axis:

$$h(x, t = 0) = wx \theta(-x) - w'x \theta(x) \quad (21)$$

where $\theta(x)$ is the Heaviside step function. In this paper we focus on the limit $w' \rightarrow +\infty$. In terms of the DP it corresponds to a (left) half-space problem with partition sum

$$h_w(x, t) = \ln Z_w(x, t) \quad (22)$$

$$Z_w(x, t) = \int_{-\infty}^0 dy e^{wy} Z(x, t|y, 0), \quad (23)$$

with $Z_w(x, t = 0) = \theta(-x)e^{wx}$. Hence for $w = 0$ it can be seen as a "half-flat" initial condition, see Fig. 1.

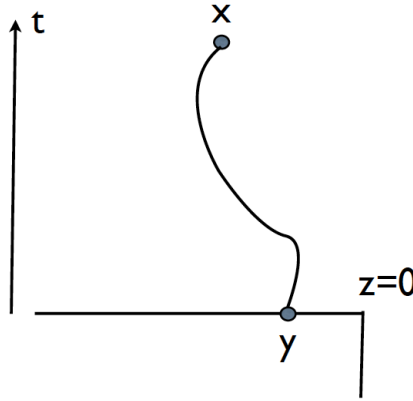


Figure 1. Half-flat initial conditions: one end-point of the DP is fixed at x , the other free on the half-line $y < z = 0$. In addition there is a weight e^{wy} which, from the STS symmetry, amounts to tilt the half-line (downward at $x = -\infty$ for $w > 0$).

Our aim here is to calculate the one-point probability distribution (PDF) of (minus) the free energy of the DP, equivalently, of the height field of the KPZ equation. We

write:

$$-F = h = \ln Z = v_0 t + \lambda \xi_t \quad , \quad Z \equiv Z_w(x, t) \quad , \quad \lambda = (\bar{c}^2 t / 4)^{1/3} \quad (24)$$

which defines ξ_t . This is done by calculating via the Bethe ansatz the integer moments of the partition function $\overline{Z^n}$. Using the statistical tilt symmetry (STS) of the problem it is easy to show the exact relation for integer moments (see e.g. [19] Appendix A):

$$\overline{Z_w^n(x, t)} = e^{-\frac{n x^2}{4t}} \overline{Z_{w+\frac{x}{2t}}^n(0, t)} \quad (25)$$

equivalently:

$$\ln Z_w(x, t) + \frac{x^2}{4t} \equiv_{\text{in law}} \ln Z_{w+\frac{x}{2t}}(0, t) \quad (26)$$

Hence, up to a simple additive piece in the free energy, the problem depends only on the combination $w + \frac{x}{2t}$, i.e. changing the endpoint is the same as changing w . Since we know that $h_0(-\infty, t) \equiv h_{\text{flat}}(0, t)$ and $h_{+\infty}(0, t) \equiv h_{\text{drop}}(0, t)$ there are thus two limits:

$$w + \frac{x}{2t} \rightarrow +\infty \quad , \quad h_{w+\frac{x}{2t}}(0, t) \equiv h_{\text{drop}}(0, t) \quad (27)$$

$$w + \frac{x}{2t} \rightarrow -\infty \quad , \quad h_{w+\frac{x}{2t}}(0, t) \equiv h_{\text{flat}}(0, t) \quad (28)$$

hence the present problem extrapolates from droplet to flat initial conditions.

4.2. relation to the $\mathcal{A}_{2 \rightarrow 1}$ transition process

Although this crossover has not yet been fully studied for the KPZ equation, it has been studied in Ref. [39] for the TASEP with an initial condition where particles are placed at the even integers, the equivalent of the "half-flat" initial condition. There the transition process $\mathcal{A}_{2 \rightarrow 1}(u)$ was defined. It enjoys the two limits ||

$$\lim_{u_1 \rightarrow +\infty} \mathcal{A}_{2 \rightarrow 1}(u + u_1) = 2^{1/3} \mathcal{A}_1(2^{-2/3} u) \quad (29)$$

$$\lim_{u_1 \rightarrow -\infty} \mathcal{A}_{2 \rightarrow 1}(u + u_1) = \mathcal{A}_2(u) \quad (30)$$

It was later shown [38] that this transition process satisfies:

$$\max_{v < u} (\mathcal{A}_2(v) - v^2) + \min(0, u)^2 = \mathcal{A}_{2 \rightarrow 1}(u) \quad (31)$$

It is then clear that the crossover studied here in the KPZ equation should be related to this transition process in the limit of infinite time. Indeed, let us generalize slightly our notations and define:

$$h_{zw}^L(x, t) = \ln \int_{y < z} dy e^{h(x, t|y, 0) + wy} \quad (32)$$

|| There we follow Ref. [38].

so that $h_w(x, t) = h_{0w}^L(x, t)$. Then we again expect that in the large time limit this solution is related to the maximum of the droplet solution on a half-line ¶

$$\begin{aligned} h_{z0}^L(x, t) &= \ln \int_{y < z} dy e^{h(x, t|y, 0)} \equiv \ln \int_{y < z} dy e^{h(y, t|x, 0)} \\ &\simeq_{t \rightarrow \infty} \max_{y < z} h(y, t|x, 0) \equiv \max_{y < z-x} h_{drop}(y, t) \end{aligned} \quad (33)$$

From (15) one then sees that (in units $\bar{c} = 1$):

$$h_z^L(x, t) \simeq_{t \rightarrow \infty} t^{1/3} \max_{v < \frac{z-x}{2t^{2/3}}} (\mathcal{A}_2(v) - v^2) = t^{1/3} [\mathcal{A}_{2 \rightarrow 1}(u) - \min(0, u)^2]_{u = \frac{z-x}{2t^{2/3}}} \quad (34)$$

More generally:

$$h_{zw}^L(x, t) \simeq_{t \rightarrow \infty} \max_{y < z} (h(y, t|x, 0) + wy) \equiv wx + \max_{y < z-x} (h_{drop}(y, t) + wy) \quad (35)$$

$$= wx + t^{1/3} \max_{2t^{2/3}v < z-x} (-v^2 + \mathcal{A}_2(v) + 2^{2/3}\tilde{w} \times 2v) \quad (36)$$

$$\equiv -\frac{x^2}{4t} + t(w + \frac{x}{2t})^2 + \max_{v < u} (-v^2 + \mathcal{A}_2(v)) \quad (37)$$

$$= -\frac{x^2}{4t} + t(w + \frac{x}{2t})^2 + t^{1/3} [\mathcal{A}_{2 \rightarrow 1}(u) - \min(0, u)^2]_{u = -t^{1/3}(w + \frac{x-z}{2t})} \quad (38)$$

where we have defined the scaled variable $\tilde{w} = \lambda w$ and used the stationarity of \mathcal{A}_2 to shift its argument. We also note that the last line can be rewritten as:

$$h_{zw}^L(x, t) \simeq_{t \rightarrow \infty} h_{zw}^{L0}(x, t) + t^{1/3} \mathcal{A}_{2 \rightarrow 1}(u) \quad (39)$$

where $h_{zw}^{(L0)}(x, t) = \max_{y < z} (-\frac{(y-x)^2}{4t} + wy)$ is the large time solution of the KPZ equation with the same initial conditions, in the absence of noise. This shows why the term $-\min(0, u)^2$ has been included in the definition of the transition Airy process chosen in Ref. [39].

4.3. generating function and its large time limit

To later extract the PDF from the moments we now introduce the generating function, as in our previous works [16, 19, 20]:

$$g_\lambda(s) = \overline{e^{-\lambda s} Z_w(x, t)} = 1 + \sum_{n=1}^{\infty} \frac{(-e^{-\lambda s})^n}{n!} \overline{Z_w(x, t)^n} = \overline{\exp(-e^{-\lambda(s-\xi)})} \quad (40)$$

Once $g_\lambda(s)$ is known, the PDF of (minus) the rescaled free energy, $P(\xi)$, at large time (i.e. $\lambda \rightarrow \infty$) is immediately extracted as:

$$g_\infty(s) = \lim_{\lambda \rightarrow \infty} g_\lambda(s) = \overline{\theta(s - \xi)} = \text{Prob}(\xi < s). \quad (41)$$

¶ for convenience we drop here and below the constant $v_0 t$, which is easily restored.

Here we will calculate the generating function $g_{+\infty}(s)$ for the half-flat initial condition using the replica Bethe Ansatz. One will check that it does reproduce correctly the two limits:

$$w + \frac{x}{2t} \rightarrow +\infty \quad , \quad g_{\infty}(s) = \lim_{t \rightarrow \infty} \text{Prob}(\xi_t < s) = F_2(2^{-2/3}s) \quad (42)$$

$$w + \frac{x}{2t} \rightarrow -\infty \quad , \quad g_{\infty}(s) = \lim_{t \rightarrow \infty} \text{Prob}(\xi_t < s) = F_1(s) \quad (43)$$

in terms of the scaled random variable ξ_t defined in (24), where $F_1(s)$ and $F_2(s)$ are respectively the GOE and GUE Tracy Widom distributions.

5. Quantum mechanics and Bethe Ansatz

The calculation of the n -th integer moment of the DP partition sum can be expressed [41, 42] using the eigenstates Ψ_{μ} and eigenenergies E_{μ} of the *attractive* Lieb-Liniger Hamiltonian for n bosons [18]:

$$H_n = - \sum_{\alpha=1}^n \frac{\partial^2}{\partial x_{\alpha}^2} - 2\bar{c} \sum_{1 \leq \alpha < \beta \leq n} \delta(x_{\alpha} - x_{\beta}). \quad (44)$$

namely ⁺ [19]:

$$\overline{Z_w(x, t)^n} = \sum_{\mu} \frac{\Psi_{\mu}^*(x, \dots x)}{||\mu||^2} e^{-tE_{\mu}} \left(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_{\alpha} e^{wy_{\alpha}} \right) \Psi_{\mu}(y_1 \dots y_n). \quad (45)$$

These eigenstates are known from the Bethe ansatz [18]. They are parameterized by a set of rapidities $\mu \equiv \{\lambda_1, \dots \lambda_n\}$ which are solution of a set of coupled equations, the Bethe equations (see below). They take the (un-normalized) form (totally symmetric in the x_{α}):

$$\Psi_{\mu}(x_1, \dots x_n) = \sum_P A_P \prod_{j=1}^n e^{i \sum_{\alpha=1}^n \lambda_{P_{\alpha}} x_{\alpha}}, \quad A_P = \prod_{n \geq \beta > \alpha \geq 1} \left(1 + \frac{i\bar{c} \text{sgn}(x_{\beta} - x_{\alpha})}{\lambda_{P_{\beta}} - \lambda_{P_{\alpha}}} \right). \quad (46)$$

where the sum runs over all $n!$ permutations P of the rapidities λ_j . The corresponding eigenenergies are $E_{\mu} = \sum_{\alpha=1}^n \lambda_{\alpha}^2$. In the formula (45) we also need:

$$\Psi_{\mu}^*(x, \dots x) = n! e^{-ix \sum_{\alpha} \lambda_{\alpha}}. \quad (47)$$

Before discussing the last two ingredients in (45), i.e. the norms $||\mu||^2$ and the half-space integrals of the Bethe eigenfunctions, let us recall the spectrum of H_n in the limit of infinite system size, i.e. the rapidities solution to the Bethe equations [43]. A general eigenstate is built by partitioning the n particles into a set of $n_s \leq n$ bound

⁺ for convenience we take the complex conjugate, since the total expression is real it is immaterial

states called *strings* formed by $m_j \geq 1$ particles with $n = \sum_{j=1}^{n_s} m_j$. The rapidities associated to these states are written as

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(m_j + 1 - 2a) \quad (48)$$

Here, $a = 1, \dots, m_j$ labels the rapidities within the string $j = 1, \dots, n_s$. Inserting these in (46) leads to the Bethe eigenstates of the infinite system, and their corresponding eigen-energies:

$$E_\mu = \sum_{j=1}^{n_s} m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1). \quad (49)$$

For now on, we use units where $\bar{c} = 1$. The calculation of the norms is involved. The result however is simple: in the large system size L limit it can be written as [44]:

$$\frac{1}{\|\mu\|^2} = \frac{1}{n! L^{n_s}} \prod_{1 \leq i < j \leq n_s} \Phi_{k_i, m_i, k_j, m_j} \prod_{j=1}^{n_s} \frac{1}{m_j^2}, \quad \Phi_{k_i, m_i, k_j, m_j} = \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \quad (50)$$

Consider now the integral in (45) over the negative half-space. Since the wave functions are totally symmetric in their arguments, it can be performed on the sector $y_1 < y_2 < \dots < y_n < 0$. Using that:

$$G_\lambda = \int_{-\infty < y_1 < y_2 < \dots < y_n < 0} dy_1 \dots dy_n e^{\sum_{\alpha=1}^n (w + i\lambda_\alpha) y_\alpha} = \prod_{j=1}^n \frac{1}{jw + i\lambda_1 + \dots + i\lambda_j},$$

a "miracle" occurs upon performing the summation over the permutations, leading to the factorized form [19, 20]:

$$\begin{aligned} \left(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_\alpha e^{wy_\alpha} \right) \Psi_\mu(y_1 \dots y_n) &= \sum_P A_P G_{P\lambda} \\ &= \frac{n!}{\prod_{\alpha=1}^n (w + i\lambda_\alpha)} \prod_{1 \leq \alpha < \beta \leq n} \frac{2w + i\lambda_\alpha + i\lambda_\beta - 1}{2w + i\lambda_\alpha + i\lambda_\beta}. \end{aligned} \quad (51)$$

Let us now explain a point which was implicit in [19, 20]. Note that, strictly, (51) is valid only for $\text{Re}(jw + i\lambda_1 + \dots + i\lambda_j) > 0$ for all j . Hence (51), which involves all $G_{P\lambda}$, is valid a priori when all λ_j are real and $w > 0$. However, one easily sees that its validity is more general. An important property of Ψ_μ for a string state is that $A_P = 0$ (for $x_1 < \dots < x_n$) unless the imaginary parts of the rapidities belonging to a given string are in increasing order. For instance for $n_2 = 2, m_1 = 2, m_2 = 3$, A_P is non-zero only for $(\lambda_1, \dots, \lambda_5)$ obtained from $(k_1 - \frac{i\bar{c}}{2}, k_1 + \frac{i\bar{c}}{2}, k_2 - \frac{3i\bar{c}}{2}, k_2 - \frac{i\bar{c}}{2}, k_2 + \frac{i\bar{c}}{2}, k_2 + \frac{3i\bar{c}}{2})$ by a permutation of S_5 which respects the order of the imaginary parts inside each string (e.g. $k_1 - \frac{i\bar{c}}{2}$ must always appear before $k_1 + \frac{i\bar{c}}{2}$). From this it is easy to see that the condition $\text{Re}(jw + i\lambda_{P_1} + \dots + i\lambda_{P_j}) > 0$ for all j is satisfied for all terms with non zero A_P . Physically it just expresses the fact that the bound states have a convergent

integral over space, and $w > 0$ is needed only to make the integral over the center of mass convergent.

6. Previous result: starting formula for the generating function

Let us recall the derivation in Ref. [19, 20], as it will be needed for further generalizations. From the ingredients (47), (49), (50), (51) we can now perform the summation \sum_μ over the eigenstates in (45). It factors into a sum over the string variables k_j, m_j . One shows that in the infinite system the string momenta $m_j k_j$ are quantized as free particles, hence we can replace $\sum_{k_j} \rightarrow m_j L \int \frac{dk_j}{2\pi}$, all factors L cancel with the norms. One obtains the moments \overline{Z}^n as a sum $(m_1, \dots, m_{n_s})_n$ over all the partitioning of n :

$$\overline{Z}^n = \sum_{n_s=1}^n \frac{n! 2^n}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \prod_{j=1}^{n_s} \int \frac{dk_j}{2\pi} \frac{1}{m_j} e^{m_j^3 \frac{t}{12} - m_j k_j^2 t - i x m_j k_j} \Phi[k, m] S^w[k, m] D^w[k, m] \quad (52)$$

together with the generating function as an expansion in the number of strings:

$$g_\lambda(s) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, s) \quad (53)$$

$$Z(n_s, s) = \sum_{m_1, \dots, m_{n_s}=1}^{\infty} \prod_{j=1}^{n_s} \frac{2^{m_j}}{m_j} \int \frac{dk_j}{2\pi} S_{m_j, k_j}^w e^{m_j^3 \frac{t}{12} - m_j k_j^2 t - \lambda m_j s - i x m_j k_j} \prod_{1 \leq i < j \leq n_s} \tilde{D}_{m_i, k_i, m_j, k_j}^w \quad (54)$$

In $g_\lambda(s)$ the summations over the m_j are free. The factors S^w and D^w are obtained by inserting the string rapidities (48) into (51). They read:

$$S_{m, k}^w = \frac{(-1)^m \Gamma(z)}{\Gamma(z + m)} \quad , \quad z = 2ik + 2w. \quad (55)$$

and

$$D_{m_1, k_1, m_2, k_2}^w = \frac{\Gamma(1 - z - \frac{m_1 + m_2}{2}) \Gamma(1 - z + \frac{m_1 + m_2}{2})}{\Gamma(1 - z + \frac{m_1 - m_2}{2}) \Gamma(1 - z - \frac{m_1 - m_2}{2})} \quad (56)$$

$$= (-1)^{m_2} \frac{\Gamma(1 - z + \frac{m_1 + m_2}{2}) \Gamma(z + \frac{m_1 - m_2}{2})}{\Gamma(1 - z + \frac{m_1 - m_2}{2}) \Gamma(z + \frac{m_1 + m_2}{2})} \quad , \quad z = ik_1 + ik_2 + 2w. \quad (57)$$

and in (52), (54) we have defined the notations:

$$\tilde{D}_{m_i, k_i, m_j, k_j}^w = D_{m_i, k_i, m_j, k_j}^w \Phi_{k_i, m_i, k_j, m_j} \quad , \quad S^w[k, m] = \prod_{j=1}^{n_s} S_{m_j, k_j}^w \quad (58)$$

$$D^w[k, m] = \prod_{1 \leq i < j \leq n_s} D_{m_i, k_i, m_j, k_j}^w \quad , \quad \Phi[k, m] = \prod_{1 \leq i < j \leq n_s} \Phi_{k_i, m_i, k_j, m_j} \quad (59)$$

involving the factor (50) coming from the norm. Note that we have also performed the shift $Z \rightarrow Z e^{\bar{c}^2 t / 12}$ which yields the factor $-\bar{c}^2 / 12$ in v_0 in Section 3.

Two remarks are in order:

(i) The STS relation (25) between moments can be retrieved by performing the change of integration variable in the integral (54):

$$ik_j \rightarrow ik_j + \frac{x}{2t} \quad (60)$$

resulting in the global shift $Z(n_s, s)|_{x,w} \rightarrow Z(n_s, s)|_{0,w'} e^{-\frac{x^2}{4t} \sum_j m_j}$ with a new value $w \rightarrow w' = w + \frac{x}{2t}$. Since we know that this STS relation holds this shift (followed by shifting the integration contour back to the real axis) must be legitimate provided $w' = w + \frac{x}{2t} > 0$ since this is the assumption used to derive (54).

(ii) while the formula for the moments $\overline{Z^n}$ is well defined, because of the exponential cubic divergence of the series, the formula (54) should be taken in some analytical continuation sense, i.e. it is valid as a formal series in t . One way to do that, as discussed in [16, 17] and below, is to use the *Airy trick*, valid for $\text{Re}(z) > 0$:

$$\int_{-\infty}^{\infty} dy \text{Ai}(y) e^{yz} = e^{z^3/3}. \quad (61)$$

the summations over m being then carried later at fixed y and usually convergent.

7. Rescaling and Airy trick

Let us first perform some rescaling and rearrangement of our starting expression, to make easier the large time limit in the next Section. In the $\bar{c} = 1$ dimensionless units one has $t = 4\lambda^3$. One can then define the scaled position and slope:

$$w = \frac{\tilde{w}}{\lambda} = \frac{\tilde{w}}{(t/4)^{1/3}} \quad , \quad x = \lambda^2 \tilde{x} = (t/4)^{2/3} \tilde{x} \quad (62)$$

such that \tilde{w} and \tilde{x} will be kept finite in the large time limit. One then perform the change $k_j \rightarrow k_j/\lambda$ in the integral. We get:

$$\begin{aligned} Z(n_s, s) &= \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} \frac{2^{m_j}}{\lambda m_j} \int \frac{dk_j}{2\pi} \frac{(-1)^{m_j} \Gamma(\frac{2ik_j + 2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik_j + 2\tilde{w}}{\lambda} + m_j)} e^{\frac{1}{3}\lambda^3 m_j^3 - 4\lambda m_j k_j^2 - \lambda m_j s - i\lambda m_j k_j \tilde{x}} \\ &\times \prod_{1 \leq i < j \leq n_s} D_{m_i, k_i/\lambda, m_j, k_j/\lambda}^w \frac{4(k_i - k_j)^2 + \lambda^2(m_i - m_j)^2}{4(k_i - k_j)^2 + \lambda^2(m_i + m_j)^2}. \end{aligned} \quad (63)$$

An equivalent expression is obtained using the Airy trick and the double Cauchy identity:

$$\prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + \lambda^2(m_i - m_j)^2}{4(k_i - k_j)^2 + \lambda^2(m_i + m_j)^2} = \det \left[\frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j} \right]_{n_s \times n_s} \prod_{j=1}^{n_s} (2\lambda m_j)$$

followed by the shift $y_j \rightarrow y_j + s + 4k_j^2$, which leads to:

$$Z(n_s, s) = \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} 2^{m_j+1} \int \frac{dk_j}{2\pi} dy_j \frac{(-1)^{m_j} \Gamma(\frac{2ik_j+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda} + m_j)} Ai(y_j + 4k_j^2 + s) e^{\lambda m_j (y_j - ik_j \tilde{x})} \\ \times \prod_{1 \leq i < j \leq n_s} D_{m_i, k_i/\lambda, m_j, k_j/\lambda}^w \times \det \left[\frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j} \right]_{n_s \times n_s} \quad (64)$$

Note that we know from the STS relation (26) that $g(s)$ only depends on the following combination of variables:

$$g_\lambda(s; w, x) = \tilde{g}_\lambda(s + \frac{\tilde{x}^2}{16}, \tilde{w} + \frac{\tilde{x}}{8}) \quad (65)$$

8. Large time limit and Fredholm determinant form

We now study the limit of large time, i.e. $\lambda \rightarrow +\infty$. We will now *assume* that in this limit we can set the complicated factor $D_{m_i, k_i/\lambda, m_j, k_j/\lambda}^w \rightarrow 1$, which we do from now on. This appears to be true *a posteriori* from the result we will obtain. An attempt at a justification is discussed in Appendix A.

Let us evaluate the resulting expression setting $D_{m_i, k_i/\lambda, m_j, k_j/\lambda}^w \rightarrow 1$ in (64), keeping for now arbitrary λ and keeping in mind that ultimately we will be interested in the limit $\lambda \rightarrow +\infty$. Using the expression of a determinant as a sum of permutations $\sigma \in S_{n_s}$ of signature $(-1)^\sigma$, $\det M = \sum_\sigma (-1)^\sigma \prod_{j=1}^{n_s} M_{j, \sigma(j)}$, and the reexponentiation formula $\frac{1}{a} = \int_{v>0} e^{-av}$ we introduce n_s auxiliary variables v_j and rewrite:

$$Z(n_s, s) = \sum_\sigma (-1)^\sigma \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} 2^{m_j+1} \int_{v_j>0} \int \frac{dk_j}{2\pi} dy_j \frac{(-1)^{m_j} \Gamma(\frac{2ik_j+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda} + m_j)} Ai(y_j + 4k_j^2 + s) \\ \times e^{\lambda m_j (y_j - ik_j \tilde{x}) - 2i(k_j - k_{\sigma(j)}) v_j - v_j (m_j + m_{\sigma(j)})} \quad (66)$$

We can now use $\sum_j v_j a_{\sigma(j)} = \sum_j v_{\sigma^{-1}(j)} a_j$ and relabel the sum over permutations as $\sigma \rightarrow \sigma^{-1}$ to obtain:

$$Z(n_s, s) = \sum_\sigma (-1)^\sigma \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} 2^{m_j+1} \int_{v_j>0} \int \frac{dk_j}{2\pi} dy_j \frac{(-1)^{m_j} \Gamma(\frac{2ik_j+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda} + m_j)} Ai(y_j + 4k_j^2 + s) \\ \times e^{\lambda m_j (y_j - ik_j \tilde{x}) - 2ik_j (v_j - v_{\sigma(j)}) - m_j (v_j + v_{\sigma(j)})} \quad (67)$$

where now we can further shift the variable $y \rightarrow y + v_j + v_{\sigma(j)} + ik_j \tilde{x}$ using the identity (for $\lambda > 0$):

$$\int_{-\infty}^{+\infty} dy Ai(y) e^{\lambda(y - ikx)} = \int_{-\infty}^{+\infty} dy Ai(y + ikx) e^{\lambda y} \quad (68)$$

Under this form it clearly appears that $Z(n_s, s)$ has a determinantal form:

$$Z(n_s, s) = \prod_{j=1}^{n_s} \int_{v_j>0} \det M(v_i, v_j) |_{n_s \times n_s} \quad (69)$$

with the Kernel:

$$M(v_1, v_2) = \int \frac{dk}{2\pi} dy Ai(y + 4k^2 + ik\tilde{x} + v_1 + v_2 + s) e^{-2ik(v_1 - v_2)} \phi_\lambda(k, y) \quad (70)$$

$$\phi_\lambda(k, y) = \sum_{m=1}^{\infty} \frac{(-1)^m 2^{m+1} \Gamma(\frac{2ik+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik+2\tilde{w}}{\lambda} + m)} e^{\lambda my} \quad (71)$$

which we now study in the large λ limit. We show that:

$$\phi_{+\infty}(k, y) = -2\theta(y) - \frac{1}{ik + \tilde{w}} \delta(y) \quad (72)$$

There are several ways to do that, the simplest is to use the Mellin-Barnes (MB) identity:

$$\sum_{m=1}^{+\infty} (-1)^m f(m) = \frac{-1}{2i} \int_C ds \frac{1}{\sin \pi s} f(s) \quad (73)$$

where $C_j = a + ir$, $0 < a < 1$, $r \in]-\infty, \infty[$. It requires being able to deform the contour and close it around the positive real axis, picking up the residues of the inverse sine function. Clearly the conditions for that are (i) $f(s)$ has no pole for $Re(s) \geq a$ (ii) $f(s)$ does not grow too fast at large $Re(s) > 0$. The first condition is necessary for us not to "miss" any pole: in fact if there are such poles in $f(s)$ they can just be added to the formula. Here both conditions are clearly satisfied. Hence we obtain:

$$\phi_\lambda(k, y) = \frac{-1}{2i} \int_C ds \frac{1}{\sin \pi s} 2^{s+1} \frac{\Gamma(\frac{2ik+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik+2\tilde{w}}{\lambda} + s)} e^{\lambda sy} \quad (74)$$

rescaling $s_j \rightarrow s_j/\lambda$, we obtain, in the limit $\lambda \rightarrow +\infty$:

$$\phi_{+\infty}(k, y) = \int_C \frac{-ds}{2i\pi s} \frac{2ik + 2\tilde{w} + s}{ik + \tilde{w}} e^{sy} \quad (75)$$

where now the contours C_j are the same as above, but with $a = 0^+$. We used that $\Gamma(x/\lambda) \sim \lambda/x$ at large λ . We can now perform the integrals over the s_j , using:

$$\int_C \frac{-ds}{2i\pi s} e^{sy} = -\theta(y) \quad , \quad \int_C \frac{-ds}{2i\pi} e^{sy} = -\delta(y) \quad (76)$$

and we obtain (72). Note that "undoing" the MB trick one sees that at large λ :

$$\phi_\lambda(k, y) \simeq \sum_{m=1}^{\infty} (-1)^m \frac{2ik + 2\tilde{w} + \lambda m}{ik + \tilde{w}} e^{\lambda my} \quad (77)$$

$$= (2 + \frac{1}{ik + \tilde{w}} \partial_y) \frac{e^{\lambda y}}{1 + e^{\lambda y}} \xrightarrow{\lambda \rightarrow +\infty} -2\theta(y) - \frac{1}{ik + \tilde{w}} \delta(y) \quad (78)$$

re-obtaining the same result via a direct summation over m *

* Note that summation over m of for $\phi_\lambda(k, y)$ is also possible at any λ leading to an hypergeometric function, see Appendix E.4 of [20].

Inserting (72) into the Kernel M we finally obtain the large λ limit:

$$Z(n_s, s) = \prod_{j=1}^{n_s} (-1)^{n_s} \int_{v_j > 0} \det K(v_i, v_j) |_{n_s \times n_s} \quad (79)$$

Hence the generating function takes the form of a Fredholm determinant:

$$g_\infty(s) = \text{Det}[I - \mathcal{K}] \quad , \quad \mathcal{K}(v_1, v_2) = \theta(v_1)\theta(v_2)K(v_1, v_2) \quad (80)$$

with the Kernel:

$$K(v_1, v_2) = \int \frac{dk}{2\pi} dy (2\theta(y) + \frac{\delta(y)}{ik + \tilde{w}}) Ai(y + 4k^2 + ik\tilde{x} + v_1 + v_2 + s) e^{-2ik(v_1 - v_2)} \quad (81)$$

which is our main result, with here $\tilde{w} > 0$, and we recall $\tilde{x} = x/\lambda^2$ and $\tilde{w} = \lambda w$. The STS symmetry can be checked by performing a shift $ik \rightarrow ik + \frac{\tilde{x}}{8}$ and bringing back the contour to the real axis. This shifts $w \rightarrow w + \frac{\tilde{x}}{8}$ and $s \rightarrow s + \frac{\tilde{x}^2}{16}$ in agreement with (65) [46]. Hence \tilde{g}_∞ in that formula is also given by (80) and the Kernel K but with \tilde{x} set to 0.

The GUE limit is easy to check on this form for K . Setting $\tilde{x} = 0$ and $\tilde{w} \rightarrow +\infty$ the second part of K vanishes and one recovers exactly the GUE Kernel in the form given in the Eq. (26) of [16] (after the change $k \rightarrow k/2$).

To check the GOE limit is more delicate. One can take $w = 0^+$ and let $x \rightarrow -\infty$. In that limit it turns out that one can replace:

$$\frac{1}{ik + 0^+} \rightarrow 2\pi\delta(k) \quad (82)$$

(a related property was noted in [20]), and that the first term vanishes. Hence $K \rightarrow B_s$ the kernel of the F_1 distribution.

We now give an equivalent form of the Kernel where the limits can be conveniently studied.

9. Equivalent forms for the Kernel

We now display three useful identity involving Airy functions. The first one is:

$$2 \int \frac{dk}{2\pi} Ai(4k^2 + a + b + ik\tilde{x}) e^{2ik(b-a)} = 2^{-\frac{1}{3}} Ai(2^{\frac{1}{3}}(a + \frac{\tilde{x}^2}{32})) Ai(2^{\frac{1}{3}}(b + \frac{\tilde{x}^2}{32})) e^{\frac{\tilde{x}}{4}(b-a)} \quad (83)$$

where a, b, k, \tilde{x} are real numbers. This identity is well known in the case $\tilde{x} = 0$ [47]. Starting from the identity for $\tilde{x} = 0$ we can shift $ik \rightarrow ik - \frac{\tilde{x}}{8}$ and $a \rightarrow a + \frac{\tilde{x}^2}{32}$, $b \rightarrow b + \frac{\tilde{x}^2}{32}$ to obtain the above identity, but now k is on a contour parallel to the real axis. Bringing back this contour to the real line, we obtain (83).

The second identity reads, for $\tilde{w} > 0$:

$$\begin{aligned} 2 \int \frac{dk}{2\pi} Ai(4k^2 + a + b + ik\tilde{x}) \frac{e^{2ik(b-a)}}{ik + \tilde{w}} \\ = \int_0^{+\infty} dr 2^{-\frac{1}{3}} Ai(2^{\frac{1}{3}}(a + \frac{r}{4} + \frac{\tilde{x}^2}{32})) Ai(2^{\frac{1}{3}}(b - \frac{r}{4} + \frac{\tilde{x}^2}{32})) e^{\frac{\tilde{x}}{4}(b-a) - r(\frac{\tilde{x}}{8} + \tilde{w})} \end{aligned} \quad (84)$$

It is obtained by introducing an auxiliary variable writing $\frac{1}{ik+\tilde{w}} = \int_0^{+\infty} dr e^{-r(ik+\tilde{w})}$. Then using the above identity (83) with $a \rightarrow a + \frac{r}{4}$, $b \rightarrow b - \frac{r}{4}$. \sharp

Note that if we set $\tilde{w} < 0$ we can use instead $\frac{1}{ik+\tilde{w}} = -\int_{-\infty}^0 dr e^{-r(ik+\tilde{w})}$ hence we obtain exactly the same integral with $\int_0^{+\infty} dr \rightarrow -\int_{-\infty}^0 dr$. Here we do not need $\tilde{w} < 0$ however it is useful to consider the limit $\tilde{w} = 0^+$ minus $\tilde{w} = 0^-$, which picks up the residue of the pole at $k = 0$ leading to another useful identity:

$$\int_{-\infty}^{+\infty} dr 2^{-4/3} Ai(2^{\frac{1}{3}}(a + \frac{r}{4} + \frac{\tilde{x}^2}{32})) Ai(2^{\frac{1}{3}}(b - \frac{r}{4} + \frac{\tilde{x}^2}{32})) e^{-r\frac{\tilde{x}}{8}} = Ai(a+b) e^{\frac{\tilde{x}}{4}(a-b)} \quad (85)$$

Using (83) and (84) with $a = v_1 + \frac{y+s}{2}$, $b = v_2 + \frac{y+s}{2}$ we can rewrite the Kernel (81) equivalently [46] as:

$$K(v_1, v_2) = \int_0^{+\infty} dr \int \frac{dk}{2\pi} dy (\theta(y)\delta(r) + \frac{1}{2}\delta(y)) \quad (86)$$

$$\times 2^{-1/3} Ai(2^{\frac{1}{3}}(v_1 + \frac{y+s}{2} + \frac{r}{4} + \frac{\tilde{x}^2}{32})) Ai(2^{\frac{1}{3}}(v_2 + \frac{y+s}{2} - \frac{r}{4} + \frac{\tilde{x}^2}{32})) e^{-r(\tilde{w} + \frac{\tilde{x}}{8})} \quad (87)$$

We now rescale $y \rightarrow 2^{2/3}y$, $r \rightarrow 2^{5/3}r$. Using the similarity transformation:

$$K(v_1, v_2) = 2^{\frac{1}{3}} \tilde{K}(2^{\frac{1}{3}}v_1, 2^{\frac{1}{3}}v_2) \quad (88)$$

and changing notations from $r \rightarrow y$ in the second integral, we obtain the equivalent form for the Kernel:

$$g_\infty(s) = Prob(\xi < s) = Det[I - \mathcal{K}] \quad , \quad \mathcal{K}(v_1, v_2) = \theta(v_1)\theta(v_2)\tilde{K}(v_1, v_2) \quad (89)$$

$$\tilde{K}(v_1, v_2) = K_{Ai}(v_1 + \sigma, v_2 + \sigma) + K_2^u(v_1 + \sigma, v_2 + \sigma) \quad , \quad \sigma = 2^{-2/3}(s + \frac{\tilde{x}^2}{16}) \quad (90)$$

in terms of the standard Airy Kernel (14) and the Kernel:

$$\begin{aligned} K_2^u(v_1, v_2) &= \int_{y>0} dy Ai(v_1 + y) Ai(v_2 - y) e^{2yu} \quad , \quad u = -2^{2/3}(\tilde{w} + \frac{\tilde{x}}{8}) \\ &= - \int_{y<0} dy Ai(v_1 + y) Ai(v_2 - y) e^{2yu} + 2^{-\frac{1}{3}} Ai(2^{-\frac{1}{3}}(v_1 + v_2 - 2u^2)) e^{-u(v_1 - v_2)} \end{aligned} \quad (91)$$

where we recall $\tilde{w} = \lambda w$, $x = \lambda^2 \tilde{x}$ and $\lambda = (t/4)^{1/3}$. The second form for K_2 is obtained by substituting $r \rightarrow 2^{5/3}y$, $a + \frac{\tilde{x}^2}{32} \rightarrow 2^{-1/3}v_1$, $b + \frac{\tilde{x}^2}{32} \rightarrow 2^{-1/3}v_2$ and $\frac{\tilde{x}}{8} \rightarrow -2^{-2/3}u$ in (85).

\sharp Alternatively one can first shift $ik \rightarrow ik + \tilde{x}/8$ in the l.h.s, which shows that the integral depends only on the STS invariant combination $\tilde{w} + \frac{\tilde{x}}{8}$, and recover the r.h.s. However this is legitimate only if $\text{sgn}\tilde{w} = \text{sgn}(\tilde{w} + \frac{\tilde{x}}{8})$ otherwise we cross a pole and generate an additional pole contribution.

10. GUE and GOE limits and the connection to $\mathcal{A}_{2 \rightarrow 1}$

It is easy to recover the GUE droplet limit by taking $\tilde{w} + \frac{\tilde{x}}{8} \rightarrow +\infty$ in (89)-(91). One sees that K_2 vanishes in that limit, hence one recovers:

$$g_\infty(s) = \text{Prob}(\xi < s) \rightarrow F_2(\sigma = 2^{-2/3}(s + \frac{\tilde{x}^2}{16})) \Leftrightarrow \lambda\xi = \chi_2 t^{1/3} - \frac{x^2}{4t} \quad (92)$$

which coincides with the known result (15). Note that it is in fact a double limit, where we keep the combination $s + \frac{\tilde{x}^2}{16}$ finite, to account for the average profile $-\frac{x^2}{4t}$ of the droplet solution.

To recover the GOE flat limit, we need to consider $\tilde{w} + \frac{\tilde{x}}{8} \rightarrow -\infty$ (while keeping $\tilde{w} > 0$, a purely technical restriction). This time, since the average profile is flat we keep s finite. Hence $K_{Ai}(v_1 + \sigma, v_2 + \sigma)$ vanishes in that limit, and so does the first term in the second expression for K_2 . Hence we are left with only the second piece of K_2 , more precisely:

$$\mathcal{K}(v_1, v_2) \simeq_{\tilde{x} \rightarrow -\infty} 2^{-\frac{1}{3}} Ai(2^{-\frac{1}{3}}(v_1 + v_2 + 2\sigma - 2u^2)) \theta(v_1) \theta(v_2) \quad (93)$$

where we have again discarded the factor $e^{-\tau(v_1 - v_2)}$ which is immaterial in calculating the Fredholm determinant. Performing the change $v_1 \rightarrow 2^{1/3}v_1$, $v_2 \rightarrow 2^{1/3}v_2$ and comparing with (24) we obtain

$$g_\infty(s) = \text{Prob}(\xi < s) \rightarrow F_1(s - 4\tilde{w}^2 - \tilde{w}\tilde{x}) \Leftrightarrow \lambda\xi = \lambda\chi_1 + wx + tw^2 \quad (94)$$

hence we recover the GOE Tracy Widom distribution, up to a shift equal to the average profile, i.e. the solution of the KPZ equation in the absence of noise with initial condition wx .

One can now compare (89)-(91) with the result of the Appendix A of [39]. The Kernel there is identical to ours provided we identify:

$$2^{-2/3}\xi = -2^{-2/3}\frac{\tilde{x}^2}{16} + \mathcal{A}_{2 \rightarrow 1}(u) + \max(0, u)^2 \quad , \quad u = -2^{2/3}(\tilde{w} + \frac{\tilde{x}}{8}) \quad (95)$$

This can also be written as (ξ being defined in (24)):

$$\lambda\xi = -\frac{x^2}{4t} + t^{1/3}(\mathcal{A}_{2 \rightarrow 1}(u) - \min(0, u)^2 + u^2) \quad (96)$$

which is exactly (in rescaled variables, and for $z = 0$) the prediction of Eq. (38) obtained there from the argument of dominance of the maximum in the large time limit. Hence we recover quite precisely for the KPZ equation, from the RBA calculation, the transition process which was obtained in the context of the TASEP. This shows the desired universality.

11. Generalization: maxima of the $\mathcal{A}_{2 \rightarrow 1}$ transition process

11.1. definitions and relations

It turns out that the Bethe Ansatz formula (45) is easily generalized to study the partition sums of a DP where each of the two endpoints is free to explore its own half-space. This in turn gives interesting information on the extremal properties of the transition process $\mathcal{A}_{2 \rightarrow 1}$ (minus a half-parabola). There are clearly two cases, either the two half-spaces are on the same side, or on opposite sides, see Fig. 2 and Fig. 3. We thus now define:

$$h_{zwz'w'}^{LL}(t) := \ln \int_{x < z'} dx \int_{y < z} dy e^{h(x,t|y,0) + wy + w'x} = \ln \int_{x < z'} dx e^{h_{wz}^L(x,t) + w'x} \quad (97)$$

$$h_{zwz'w'}^{LR}(t) := \ln \int_{x > z'} dx \int_{y < z} dy e^{h(x,t|y,0) + wy - w'x} = \ln \int_{x > z'} dx e^{h_{wz}^L(x,t) - w'x} \quad (98)$$

Note that h^{LR} is well defined for $w = w' = 0$, since, as can be seen in Fig. 3 the line tension of the polymer (i.e. the diffusion kernel) makes all integrals convergent. On the other hand h^{LL} is finite only for $w, w' > 0$ since there is nothing to prevent the polymer to be arbitrary far to the left, see Fig. 2.

Again we can expect that in the large t limit these integrals are dominated by their maximum. One thus expects, setting $w = w' = 0$ that:

$$h_{zz'}^{LR}(t) = \ln \int_{x > z'} dx e^{h_z^L(x,t)} \rightarrow_{t \rightarrow +\infty} \max_{x > z'} h_z^L(x,t) \quad (99)$$

$$\simeq t^{1/3} \max_{x > z'} [\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2]_{v=(z-x)/(2t^{2/3})} \quad (100)$$

$$= t^{1/3} \max_{v < u = \frac{z-z'}{2t^{2/3}}} [\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2] \quad (101)$$

where in the second line we have used (34). Because of the term $-\min(0, v)^2$ this optimization problem is well defined for $v \rightarrow -\infty$, the GUE side. Optimization on the complementary interval $[u, +\infty[$ (the GOE side) is clearly divergent.

Keeping now $w, w' > 0$ we can also write:

$$h_{zwz'w'}^{LL}(t) = \ln \int_{x < z'} dx e^{h_z^L(x,t) + w'x} \rightarrow_{t \rightarrow +\infty} \max_{x < z'} (h_{zw}^L(x,t) + w'x) \quad (102)$$

$$\simeq tw^2 + \max_{x < z'} \left((w + w')x + t^{1/3} [\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2]_{v=-t^{1/3}w + \frac{z-x}{2t^{2/3}}} \right) \quad (103)$$

$$\begin{aligned} &= -tw^2 - 2tww' + (w + w')z \\ &\quad + t^{1/3} \max_{v > u = \frac{z-z'-2tw}{2t^{2/3}}} [\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2 - 2v \times 2^{2/3}(\tilde{w} + \tilde{w}')] \end{aligned} \quad (104)$$

where in the second line we have used (38) and we require $w + w' > 0$ for the problem to be well defined. We have defined the scaled variables $\tilde{w} = \lambda w$ and $\tilde{w}' = \lambda w'$ hence $2^{2/3}(\tilde{w} + \tilde{w}') = t^{1/3}(w + w')$. Note the symmetry property which arises from

the above definitions: $h_{zwz'w'}^{LL}(t) \equiv h_{z'w'zw}^{LL}(t)$ which implies $\max_{x < z} (h_{z'w'}^L(x, t) + wx) \equiv \max_{x < z'} (h_{zw}^L(x, t) + w'x)$, hence (104) must also be symmetric w.r.t to the exchange of (z, w) with (z', w') . This can be checked in the absence of noise where the solution, in the infinite time limit, reads (this amounts to set $\mathcal{A}_{2 \rightarrow 1}(v) \rightarrow 0$ in (104)):

$$h_{zwz'w'}^{LL0}(t) = \max_{x < z', y < z} \left(-\frac{(x-y)^2}{4t} + wy + w'x \right) = \theta(z' - z > 2w't)[tw'^2 + (w + w')z] \quad (105)$$

$$+ \theta(2wt < z' - z < 2w't)[wz + w'z' - \frac{(z - z')^2}{4t}] + \theta(z' - z < -2wt)[tw^2 + (w + w')z']$$

which is complicated but clearly symmetric in the exchange of (z, w) with (z', w') . The fact that such a symmetry holds also (in law) for the noisy case is not quite easy to guess from just looking at (104). However it must be correct, and we do indeed find that our result below satisfies it.

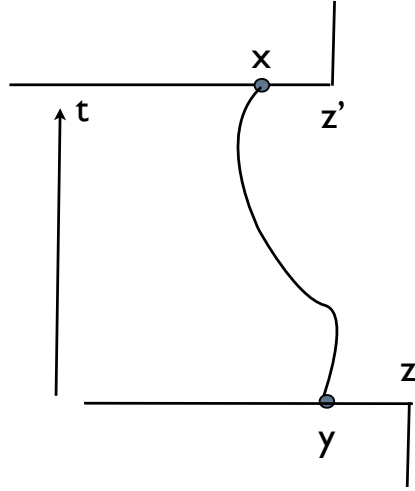


Figure 2. LL geometry: each endpoint of the DP is free on the left half-spaces $x < z'$ and $y < z$, respectively, with in addition exponential weights which amount (for $w, w' > 0$) to tilt the half-lines, $e^{w'x}$ (tilt upward at $-\infty$) and e^{wy} (tilt downward at $-\infty$).

11.2. quantum mechanics

Let us now express the moments of the partition sums:

$$Z^{LL}(t) = Z_{zwz'w'}^{LL}(t) = e^{h_{zwz'w'}^{LL}(t)} \quad (106)$$

$$Z^{LR}(t) = Z_{zwz'w'}^{LR}(t) = e^{h_{zwz'w'}^{LR}(t)} \quad (107)$$

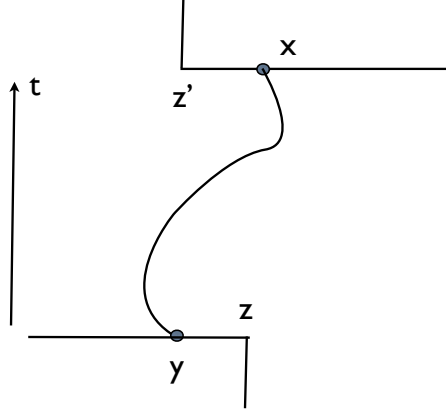


Figure 3. LR geometry: each endpoint of the DP is free on the a half-spaces $x > z'$ (right half-space) and $y < z$ (left half-space), respectively, with in addition exponential weights which amount (for $w, w' > 0$) to tilt the half-lines, $e^{-w'x}$ (tilt upward at $+\infty$) and e^{wy} (tilt downward at $-\infty$).

using the eigenstates $\mu = \{\lambda_1, \dots, \lambda_n\}$ of the Lieb-Liniger model. Generalizing formula (45) we see that:

$$\overline{(Z^{LL})^n} = \sum_{\mu} (I_{\mu}^L(w'))^* e^{(nw' - i \sum_{\alpha} \lambda_{\alpha}^*) z'} \frac{e^{-tE_{\mu}}}{||\mu||^2} e^{(nw + i \sum_{\alpha} \lambda_{\alpha}) z} I_{\mu}^L(w) \quad (108)$$

$$\overline{(Z^{LR})^n} = \sum_{\mu} (I_{\mu}^R(w'))^* e^{(-nw' - i \sum_{\alpha} \lambda_{\alpha}^*) z'} \frac{e^{-tE_{\mu}}}{||\mu||^2} e^{(nw + i \sum_{\alpha} \lambda_{\alpha}) z} I_{\mu}^L(w) \quad (109)$$

where we have defined:

$$I_{\mu}^L(w) = \prod_{j=1}^n \int_{-\infty}^0 dy_j e^{wy_j} \Psi_{\mu}(y_1 \dots y_n) \quad (110)$$

$$I_{\mu}^R(w') = \prod_{j=1}^n \int_0^{+\infty} dx_j e^{-w'x_j} \Psi_{\mu}(x_1 \dots x_n) = I_{-\mu}^L(w') \quad (111)$$

where $-\mu$ denotes the state with reversed rapidities $-\mu = \{-\lambda_1, \dots, -\lambda_n\}$ and we use $\Psi_{\mu}(-x_1, \dots, -x_n) = \Psi_{-\mu}(x_1, \dots, x_n)$, as can be seen from (46).

Up to now the formula are general. Let us now consider the limit $L \rightarrow +\infty$ and insert the string states. Let us define:

$$\ln Z^{LL} = wz + w'z' + \ln \tilde{Z}^{LL} \quad , \quad \ln Z^{LR} = wz - w'z' + \ln \tilde{Z}^{LR} \quad (112)$$

From the results of [19, 20] (Eq. (76)), and Section 6 (see notations there) we have:

$$I_{\mu}^L(w) = n!(-2)^n S^w[k, m] D^w[k, m] \quad (113)$$

$$I_{\mu}^R(w') = n!(-2)^n S^{w'}[-k, m] D^{w'}[-k, m] = I_{\mu}^L(w')^* \quad (114)$$

and we recall the property

$$(S_{m,k}^w)^* = S_{m,-k}^w \quad , \quad (D_{m_1,k_1,m_2,k_2}^w)^* = D_{m_1,-k_1,m_2,-k_2}^w \quad (115)$$

from the definitions (55), (56).

The moments thus read:

$$\overline{(\tilde{Z}^\epsilon)^n} = \sum_{n_s=1}^n \frac{n!(-4)^n}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \Phi[k, m] S^w[k, m] S^{w'}[\epsilon k, m] D^w[k, m] D^{w'}[\epsilon k, m] \quad (116)$$

$$\times \prod_{j=1}^{n_s} \frac{1}{m_j} e^{m_j^3 \frac{t}{12} - m_j k_j^2 t - i(z' - z) m_j k_j} \quad (117)$$

where from now on we denote:

$$LL \Leftrightarrow \epsilon = -1 \quad , \quad LR \Leftrightarrow \epsilon = +1 \quad (118)$$

We then define the generating functions:

$$g_\lambda^{LL}(s) = \sum_{n=0}^{+\infty} \frac{1}{n!} e^{-\lambda n s} \overline{(\tilde{Z}^{LL})^n} \quad (119)$$

and similarly for $g_\lambda^{LR}(s)$. It is easy to see that $g_\lambda^\epsilon(s)$ is then given exactly by the same formula as $g_\lambda(s)$ in (54) with the substitution [45]

$$S_{m_j, k_j}^w \rightarrow (-2)^{m_j} S_{m_j, k_j}^w S_{m_j, \epsilon k_j}^{w'} \quad , \quad x \rightarrow z' - z \quad (120)$$

$$D_{m_i, k_i, m_j, k_j}^w \rightarrow D_{m_i, k_i, m_j, k_j}^w D_{m_i, \epsilon k_i, m_j, \epsilon k_j}^{w'} \quad (121)$$

11.3. large time limit and first form for the Kernel

We now consider the large time limit and again set all factors D^w , $D^{w'}$ to unity. We define again the scaled variables $\tilde{z} = z/\lambda^2$, $\tilde{z}' = z'/\lambda^2$, $\tilde{w} = \lambda w$, $\tilde{w}' = \lambda w'$. Performing exactly the same steps as in Sections 7 and 8, and using the formula (55) we arrive at the following expressions in terms of Fredholm determinants:

$$g_\lambda^\epsilon(s) \rightarrow \text{Det}[I - \mathcal{M}] \quad , \quad \mathcal{M}(v_1, v_2) = \theta(v_1) \theta(v_2) M(v_1, v_2) \quad (122)$$

with the same Kernel M as in (70) (with $\tilde{x} = \tilde{z} - \tilde{z}'$) with:

$$\phi_\lambda(k, y) \rightarrow \phi_\lambda^\epsilon(k, y) = \sum_{m=1}^{\infty} \frac{(-1)^m 2^{2m+1} \Gamma(\frac{2ik+2\tilde{w}}{\lambda}) \Gamma(\frac{2ik\epsilon+2\tilde{w}'}{\lambda})}{\Gamma(\frac{2ik+2\tilde{w}}{\lambda} + m) \Gamma(\frac{2ik\epsilon+2\tilde{w}'}{\lambda} + m)} e^{\lambda m y} \quad (123)$$

We again use the Mellin Barnes identity (73) and rescale $s_j \rightarrow s_j/\lambda$, to obtain, in the limit $\lambda \rightarrow +\infty$:

$$\phi_{+\infty}^\epsilon(k, y) = \int_C \frac{-ds}{2i\pi s} \frac{1}{2} \frac{(2ik + 2\tilde{w} + s)(2ik\epsilon + 2\tilde{w}' + s)}{(ik + \tilde{w})(ik\epsilon + \tilde{w}')} e^{sy} \quad (124)$$

$$= \int_C \frac{-ds}{2i\pi s} \left[2 + \frac{s}{ik + \tilde{w}} + \frac{s}{ik\epsilon + \tilde{w}'} + \frac{1}{2} \frac{s^2}{(ik + \tilde{w})(ik\epsilon + \tilde{w}')} \right] e^{sy} \quad (125)$$

$$= -2\theta(y) - \left[\frac{1}{ik + \tilde{w}} + \frac{1}{ik\epsilon + \tilde{w}'} \right] \delta(y) - \frac{1}{2} \frac{1}{(ik + \tilde{w})(ik\epsilon + \tilde{w}')} \delta'(y) \quad (126)$$

Hence we finally obtain that:

$$h_{zwz'w'}^\epsilon(t) = v_0 t + wx - \epsilon w' x' + \lambda \xi \quad , \quad \lambda = (t/4)^{1/3} \quad (127)$$

with, as $t \rightarrow \infty$:

$$Prob(\xi < s) = g_\infty^\epsilon(s) = Det[I - \mathcal{K}^\epsilon] \quad , \quad \mathcal{K}^\epsilon(v_1, v_2) = \theta(v_1)\theta(v_2)K^\epsilon(v_1, v_2) \quad (128)$$

with the Kernels:

$$K^\epsilon(v_1, v_2) = \int \frac{dk}{2\pi} dy \left[2\theta(y) + \left(\frac{1}{ik + \tilde{w}} + \frac{1}{ik\epsilon + \tilde{w}'} \right) \delta(y) - \frac{1}{2} \frac{1}{(ik + \tilde{w})(ik\epsilon + \tilde{w}')} \delta(y) \partial_y \right] \\ \times Ai(y + 4k^2 + ik(\tilde{z}' - \tilde{z}) + v_1 + v_2 + s) e^{-2ik(v_1 - v_2)} \quad (129)$$

where $\tilde{w}, \tilde{w}' > 0$ and we recall that $\epsilon = -1$ for *LL* and $\epsilon = +1$ for *LR*. The STS symmetry is again recovered from the shift $ik \rightarrow ik + \frac{\tilde{z}' - \tilde{z}}{8}$ followed by the shift back to the real axis, and it shows that all the dependence in z, z', w, w' has the form [46]:

$$g_\infty^\epsilon(s) = \tilde{g}_\infty^\epsilon\left(s + \frac{(\tilde{z}' - \tilde{z})^2}{16}, \tilde{w} + \frac{\tilde{z}' - \tilde{z}}{8}, \tilde{w}' + \epsilon \frac{\tilde{z}' - \tilde{z}}{8}\right) \quad (130)$$

which is a symmetric function of its last two arguments (obvious for $\epsilon = +1$ (LR) and using $k \rightarrow -k$ followed by the transposition of the Kernel for $\epsilon = -1$ (LL)). The function $\tilde{g}_\infty^\epsilon$ has the same form (128) with the same Kernel K where $\tilde{z} - \tilde{z}'$ is set to 0.

One easily checks some limits. Taking both $\tilde{w}, \tilde{w}' \rightarrow +\infty$ in (129) one recovers the GUE Kernel. The same is thus true if one takes $\tilde{z}' - \tilde{z} \rightarrow +\infty$. If one takes $\tilde{w}' \rightarrow +\infty$ at fixed w one recovers the half-flat Kernel (89)-(91) as expected.

11.4. second form for the Kernel

We again transform these Kernels to another equivalent form. Details are given in Appendix B. Our final result is that (127) holds with:

$$g_\infty^\epsilon(s) = Prob(\xi < s) = Det[I - \mathcal{K}^\epsilon] \quad , \quad \mathcal{K}^\epsilon(v_1, v_2) = \theta(v_1)\theta(v_2)\tilde{K}^\epsilon(v_1, v_2) \quad (131)$$

with $\epsilon = -1$ for *LL*, $\epsilon = +1$ for *LR* and the Kernel [46]:

$$\tilde{K}^\epsilon(v_1, v_2) = K_{Ai}(v_1 + \sigma, v_2 + \sigma) + \bar{K}^\epsilon(v_1 + \sigma, v_2 + \sigma) \quad , \quad \sigma = 2^{-2/3}\left(s + \frac{(\tilde{z}' - \tilde{z})^2}{16}\right)$$

where K_{Ai} is the Airy Kernel (14) and:

$$\bar{K}^{LL}(v_1, v_2) = K_2^u(v_1, v_2) + K_2^{u'}(v_2, v_1) + K_3^{LL}(v_1, v_2) \quad (132)$$

$$\bar{K}^{LR}(v_1, v_2) = K_2^u(v_1, v_2) + K_2^{u'}(v_1, v_2) + K_3^{LR}(v_1, v_2) \quad (133)$$

$$u = -2^{2/3}\left(\tilde{w} + \frac{\tilde{z}' - \tilde{z}}{8}\right) \quad , \quad u' = -2^{2/3}\left(\tilde{w}' + \epsilon \frac{\tilde{z}' - \tilde{z}}{8}\right) \quad (134)$$

in terms of the Kernel K_2^u defined in (91) and in terms of a third Kernel:

$$K_3^{LL}(v_1, v_2) = (\partial_{v_1} + \partial_{v_2}) \int_0^{+\infty} dr \frac{Ai(v_1 + r)Ai(v_2 - r)e^{2ru} + Ai(v_1 - r)Ai(v_2 + r)e^{2ru'}}{2(u + u')} \\ K_3^{LR}(v_1, v_2) = (\partial_{v_1} + \partial_{v_2}) \int_0^{+\infty} dr Ai(v_1 + r)Ai(v_2 - r) \frac{e^{2ru} - e^{2ru'}}{2(u' - u)} \quad (135)$$

We note that \tilde{K}^{LR} is invariant in the exchange of u and u' , while \tilde{K}^{LL} is changed in its transpose [48] hence we recover the symmetry of the function $\tilde{g}_\infty^\epsilon$ in Eq. (130).

11.5. maximum of the transition process

As one example of application, consider $g^{LR}(s)$ and set $w = w' = 0$ hence $u = u'$. Putting together (101) together with our above result (131)-(135) and the definition of ξ (127) we obtain that:

$$t^{1/3} \max_{v < u} [\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2] = t^{1/3} \chi^u - \frac{(z - z')^2}{4t} = t^{1/3} (\chi^u - u^2) \quad (136)$$

where χ^u is distributed according to:

$$Prob(\chi^u < S) = \text{Det}[I - \mathcal{P}] \quad , \quad \mathcal{P}(v_1, v_2) = \theta(v_1)\theta(v_2)P(v_1 + S, v_2 + S) \quad (137)$$

$$P(v_1, v_2) = K_{Ai}(v_1, v_2) + (2 - \frac{1}{2}(\partial_{v_1} + \partial_{v_2})\partial_u)K_2^u(v_1, v_2) \quad (138)$$

More results concerning the maximum of $\mathcal{A}_{2 \rightarrow 1}(v) - \min(0, v)^2 - bv$ on intervals either $v \in [u, +\infty[$ (RR) or $v \in]-\infty, u]$ are also easily extracted from the above formula (in view of e.g. (104)).

12. Conclusion

To conclude we have shown how the replica Bethe Ansatz allows to find the large time limit of the PDF of the height field of the continuum KPZ equation for the half-flat initial condition. This PDF takes the form of a Fredholm determinant. We show that its Kernel can be transformed to the one found for the TASEP with the corresponding initial condition, a manifestation of KPZ universality. Being confident that the procedure for taking the large time limit gives the correct answer, although at this stage not fully justifiable, we obtain the one point PDF for the problem of a directed polymer with each endpoint on its own half-space. It also takes the form of Fredholm determinants with new Kernels that we display and analyze. These are related to the extremal statistics of the half-flat initial condition. It seems that it should now be doable, by taking a similar limit, to study the many point problem and recover the complete $\text{Airy}_{2 \rightarrow 1}$ process.

It would also be interesting to test numerically the present results, or in liquid crystal experiments. They also have consequences for the conductance g of disordered 2D conductors deep in the Anderson localized regime, specifically for the conductance from a point lead (x, t) to an extended lead $(0, y)$ $y \in [0, L[$. Extending the results of Ref. [49] we surmise that the one point distribution of $\ln g$, scaled by $t^{1/3}$ should exhibit the GUE to GOE universal crossover distribution. Similarly our LR (and LL) results can in principle be tested for two parallel half-line leads (eventually tilted by small

angles $\sim w, w'$). This could provide a rather detailed test of the conjectured relation between the positive weight DP problem and the Anderson problem, which, at present, is not based on any exactly solvable model.

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Appendix A. Large time limit

The rationale to set the D^w factor to unity in the large time limit is as follows. Assume one can use the Mellin Barnes identity (73) on the starting formula (64) and write:

$$Z(n_s, s) = \prod_{j=1}^{n_s} \int_{C_j} \frac{-ds_j}{2i\pi \sin \pi s_j} 2^{s_j+1} \int \frac{dk_j}{2\pi} \int dy_j Ai(y_j + 4k_j^2 + ik_j \tilde{x} + s) \frac{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda}) e^{\lambda s_j y}}{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda} + s_j)} \\ \times \prod_{1 \leq i < j \leq n_s} D_{s_i, k_i/\lambda, s_j, k_j/\lambda}^w \times \det \left[\frac{1}{2i(k_i - k_j) + \lambda s_i + \lambda s_j} \right]_{n_s \times n_s} \quad (\text{A.1})$$

where $C_j = a + ir_j$, $0 < a < 1$. Then rescaling $s_j \rightarrow s_j/\lambda$ and take the large λ limit one finds that indeed $D_{s_i, k_i/\lambda, s_j, k_j/\lambda}^w \rightarrow 1$ from the definition (56). Pursuing the calculation then immediately leads to the same formula as in Section 8, as the s_j integral then decouple.

This is the type of argument which was used in Ref. [27, 30, 32, 33]. At present its proper justification escapes us, however. One condition for the Mellin Barnes identity (73) to hold is that $f(s_j)$ has no poles for $\text{Res } s_j > a$. One easily sees that the determinant is not a problem as its poles for $s_i + s_j$ live on the imaginary axis. However examination of the formula (56) for $D_{s_i, k_i/\lambda, s_j, k_j/\lambda}^w$ (if we take it as the proper analytical continuation for complex m_j) shows that at fixed k_j it has numerous poles where some of the $\text{Res } s_j > a$.

Given that we know that setting $D^w \rightarrow 1$ does give the correct answer in a number of cases, including in the present study, it is quite likely that there is indeed an integral representation either identical or similar to the above one. One way to check would be to actually enumerate the additional poles, add their contributions and show that they indeed vanish for infinite λ . Another check would be to see whether the finite time solution of [19, 20] can be also retrieved from (A.1) or a modification thereof. Finally, other analytical continuations in the m_j than (56) may be searched for. These go beyond the present study. It is likely that a better understanding of why and how this limit works will come from the other routes, as limits from the ASEP, q -TASEP or the O'Connor semi-discrete polymer, which do produce better controlled nested contour integral formula (see e.g. [23, 22]).

Appendix B. Kernel manipulations

Let us start with the Kernel in (129). Writing in the last term $\frac{1}{(ik+\tilde{w})(ik\epsilon+\tilde{w}')} = \frac{1}{\tilde{w}'-\epsilon\tilde{w}}\left(\frac{1}{ik+\tilde{w}} - \frac{\epsilon}{ik\epsilon+\tilde{w}'}\right)$ using (84) with $a = v_1 + \frac{y+s}{2}$, $b = v_2 + \frac{y+s}{2}$ and rescaling $y \rightarrow 2^{2/3}y$, $r \rightarrow 2^{5/3}r$ we find:

$$K^\epsilon(v_1, v_2) = 2^{1/3} \tilde{K}^\epsilon(2^{1/3}v_1, 2^{1/3}v_2) \quad (\text{B.1})$$

Hence we obtain Eqs. (131) and (132) in the text with:

$$\bar{K}^\epsilon(v_1, v_2) = \int_0^{+\infty} dr \int \frac{dk}{2\pi} dy \delta(y) \left(1 - \frac{1}{2 \times 2^{2/3}(\tilde{w}' - \epsilon\tilde{w})} \partial_y\right) \quad (\text{B.2})$$

$$\times Ai(v_1 + y + r) Ai(v_2 + y - r) e^{-2r2^{2/3}(\tilde{w} + \frac{z'-\tilde{z}}{8})} \quad (\text{B.3})$$

$$+ \int_0^{+\infty} dr \int \frac{dk}{2\pi} dy \delta(y) \left(1 + \frac{\epsilon}{2 \times 2^{2/3}(\tilde{w}' - \epsilon\tilde{w})} \partial_y\right) \quad (\text{B.4})$$

$$\times Ai(v_1 + y + \epsilon r) Ai(v_2 + y - \epsilon r) e^{-2r2^{2/3}(\tilde{w}' + \epsilon \frac{z'-\tilde{z}}{8})} \quad (\text{B.5})$$

This then leads to the form given in the text.

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